On Representing Coalitional Games with Externalities

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ABSTRACT

We consider the issue of representing coalitional games in multi-agent systems with externalities (*i.e.*, in systems where the performance of one coalition may be affected by other co-existing coalitions). In addition to the conventional partition function game representation (PFG), we propose a number of new representations based on a new notion of externalities. In contrast to conventional game theory, our new concept is not related to the process by which the coalitions are formed, but rather to the effect that each coalition may have on the entire system and *vice versa*. We show that the new representations are *fully expressive* and, for many classes of games, more *concise* than the conventional PFG. Building upon these new representations, we propose a number of approaches to solve the *coalition structure generation* problem in systems with externalities. We show that, if externalities are characterised by various degrees of regularity, the new representations allow us to adapt coalition structure generation algorithms that were originally designed for domains with no externalities, so that they can be used when externalities are present. Finally, building upon [16] and [9], we present a unified method to solve the coalition structure generation problem in any system, with or without externalities, provided sufficient information is available.

Categories and Subject Descriptors

F.2.0 [**Analysis of Algorithms and Problem Complexity**]: General

General Terms

Theory, Algorithms, Economics

1. INTRODUCTION

Coalition formation is a fundamental issue in multi-agent system research because cooperating agents are often more efficient than individuals. Coalition games are models that capture opportunities for cooperation by explicitly modeling the ability of the agents to take joint actions as primitives [7]. To date, however, most work in this area has focused on situations (or games) with no externalities, where the performance of one coalition is independent of the performance of other coalitions that may be present in the system. Now, while such an assumption is valid for many research problems, for many others it is not. Consequently, in this paper we focus on coalitional games with externalities.

Games with externalities have been widely studied in economics and social sciences, where interdependencies between coalitions play an important role. Examples include collusion in oligopolies, where cooperating companies seek to undermine the competitive position of other firms in the market, as well as various forms of international (macroeconomic/environmental) policy coordination between countries [1][12]. For instance, when two pharmaceutical companies decide to cooperate in order to develop a new drug, all the other companies lose some of their competitive position, (*i.e.*, are subject to negative externalities). Conversly, the decision by one group of countries to reduce gas emissions may have a positive impact on other countries or regions (*i.e.*, they are subject to positive externalities).

This issue of externalities is also becoming increasingly important in domains in which multi-agent system techniques are applied. For example, the British company, Aerogistics[1], enables small- and medium-size aircraft component manufacturers and service providers to form online, *ad hoc* supply-chain coalitions to bid for manufacturing projects too large for any individual participant. Since all components must ultimately conform to the same standards, the cost of standarization procedures incurred by any coalition depends on the number and structure of other winning coalitions. As an another examples, coalitions of regional satellite mobile network operators may enable economies of scale and scope for between-region communications, and thereby "take" market share (traffic or customers) from operators outside the coalition. In response, operators outside the coalition may be able to prevent or mitigate this effect by forming appropriate coalitions. Increasingly, such wholesale telecommunications services, and the underlying coalitions supporting them, are created or provided online, and in near real-time. Non-trivial externalities may arise when coalitions are created exogenously rather than endogenously, *i.e.*, when some central authority divides agents into coalitions. In this context, a detailed analysis of an on-line retailer who groups customers in order to obtain shipment discounts is provided in Michalak *et al* [10]. Here, externalities emerge due to imperfect warehouse management, *i.e.*, if there are not enough goods to suit all coalitions, satisfying some of them makes other dissatisfied.

As demonstrated by Deng and Papadimitru [6], Sandholm and Conitzer [2, 3], Ieong and Shoham [7] and Ohta *et al* [11], the representation of a system is one of the key issues in developing efficient solutions to coalitional games. In this context, most research has focused on games with no externalities that can be modeled using a characteristic function game representation (*e.g.*, [14, 16, 17, 18]).

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[1]See `www.aerogistics.com` for more details.

The main feature of these systems is that the performance of any coalition is not affected by the way non-members are partitioned. Although characteristic function games are popular in the literature, many realistic systems cannot be modeled using this representation due to non-negligible interdependencies among the coalitions. For such systems, the *partition function game* representation, which assigns a value to every coalition in every coalition structure, has been the only available representation in the literature. In this context, the notion of externalities focuses on the way the coalitions are formed (see Section 3 for more details). While such a perspective is interesting and fruitful in many game-theoretic applications, it may become an obstacle in a multi-agent context where only the coalitions themselves matter, regardless of the process that could lead to the formation of these coalitions. Consider, for instance, the *coalition structure generation* (CSG) problem, which involves finding a *coalition structure* (*i.e.*, a division of all the agents into exhaustive and disjoint coalitions) such that the overall efficiency of the system is maximised. In this context, given a cooperative multi-agent system, the evaluation of every potential coalition structure does not need to take into consideration the process that leads to its formation. The goal of this paper is, therefore, to develop novel representations for these games. Following Ieong and Shoham [7], the representations that we present in this paper will be evaluated with respect to:

- **Expressivity:** the breadth of the class of games covered by the representation;
- **Conciseness:** the memory requirements for the representation;
- **Efficiency:** the complexity of algorithms for computing solutions using this representation;
- **Simplicity:** the ease of use of the representation.

In this context, an ideal representation should be fully expressive (*i.e.*, should cover every possible class of games), simple, as concise as possible, and allow the development of efficient algorithms. Furthermore, we will analyse **informational requirements** of every representation, *i.e.*, the extent to which information about the system is required in order to make use of a given representation.

In more detail, this work advances the state of the art by:

- Developing a new notion of externalities that is not related to the way the coalitions are formed, but rather to the direct influence of a coalition on the system and *vice versa*.
- Developing new representations (for games with externalities) that are fully expressive and have reasonable memory requirements. Furthermore, we show that, for particular classes of games with externalities, they can be reduced to significantly more compact representations compared to the conventional approach.
- Evaluating the efficiency of the new representations by considering the CSG problem in games with externalities. Specifically, we show that these representations allow for the development of efficient algorithms for solving this problem.

The remainder of the paper is organized as follows. Section 2 introduces our basic notation. Section 3 discusses the issue of externalities from a game-theoretic perspective. Section 4 introduces a novel notion of externalities, and proposes representations based on this new notion. Section 5 defines classes of games with externalities that meet particular patterns. Using those patterns, Section 6 compares the classical and our new notion of externalities and the representations stemming from them. Finally, Section 7 discusses sample applications of the new representations, before concluding in Section 8.

2. BASIC NOTATION

We will denote by $A = \{a_1, \ldots, a_n\}$ the set of all the agents in the system. A coalition structure $\Pi = \{C_1 C_2 \ldots C_m\}$ is a partition of A into coalitions. We refer to a coalition C in Π as being *embedded* in Π and denote it by (C, Π). The *set of all embedded* coalitions (in all feasible coalition structures) is denoted by $\mathcal{C}$. We denote the cardinality of any coalition C by $|C|$, and the cardinality of any coalition structure Π, *i.e.*, the number of coalitions in Π, by $|\Pi|$. For any vector $x = [x_1, \ldots, x_n]$ of size n, we denote by $\|x\|$ the sum of the elements in x, *i.e.*, $\|x\| = \sum_{i=1}^{n} x_i$. Finally, we denote by $\mathcal{S}$ the space of all coalition structures, and by $\mathcal{S}_{i_1,i_2,\ldots,i_k}$ the subspace of $\mathcal{S}$ containing all coalition structures with coalitions of sizes $i_1, i_2, \ldots, i_k$, (*cf.* [16]). For instance, in a 4-agent system, $\mathcal{S}_{3,1}$ is the subspace containing all the coalition structures that are made of two coalitions: one of size 3 and one of size 1. That is, $\{\{a_1a_2a_3\}\{a_4\}\}$, $\{\{a_1a_2a_4\}\{a_3\}\}$, $\{\{a_1a_3a_4\}\{a_2\}\}$, and $\{\{a_2a_3a_4\}\{a_1\}\}$.

3. CHARACTERISTIC VS. PARTITION FUNCTION GAMES

In this section, we briefly describe the game-theoretic approach to coalitional games without and with externalities, as well as conventional notions of externalities.

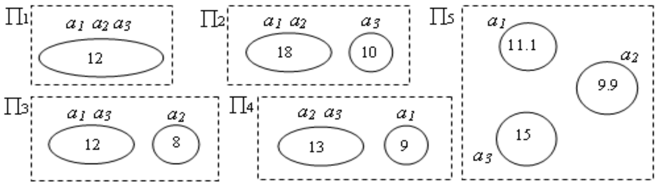
Characterist Function Game: The *characteristic function game* (CFG) representation consists of a set of agents A and a *characteristic function* v, which takes, as an input, a coalition $C \subseteq A$ and outputs its value $v(C) \in \mathbb{R}$, which reflects the performance of this coalition. In these representations, the value of any coalition is independent of any other coalition in the system.

Partition Function Game: The *partition function game* (PFG) representation, proposed in [8], consists of a set of agents A and a *partition function* $\mathcal{P}$. Specifically, for any coalition structure Π, and any coalition $C \in \Pi$, the partition function $\mathcal{P}$ outputs a value $\mathcal{P}(C; \Pi) \in \mathbb{R}$ that reflects the performance of C in Π. In contrast to $CFGs$, the value of a coalition $C \in \Pi$, as computed from a partition function, may depend upon how the other agents in the system are partitioned. In other words, this representation accounts for *externalities* in coalition formation where the performance of one coalition may be affected by the creation of other coalitions in the system. Obviously, characteristic function games are a special case of $PFGs$ where the externalities are exactly zero. This means that, in general, algorithms designed for games with no externalities cannot be directly applied to those with externalities. For instance, as mentioned in the introduction, no available CSG algorithm for characteristic function games is directly applicable to partition function games. In fact, the only CSG algorithm that accounts for non-zero externalities from coalition formation is that of [9].

For PFGs, as far as notation is concerned, a shorthand vector notation will often be used to denote all the values of embedded coalitions in a particular coalition structure. For example, the values of coalitions C_1, C_2 and C_3 in $\Pi = \{C_1C_2C_3\}$ will be denoted by a transposed vector $\mathcal{P}(\Pi) = [\mathcal{P}(C_1; \Pi), \mathcal{P}(C_2; \Pi), \mathcal{P}(C_3; \Pi)]$. An example is shown in Figure 1, where $\Pi_1 = \{\{a_1a_2a_3\}\}$, $\Pi_2 = \{\{a_1a_2\}\{a_3\}\}$, $\Pi_3 = \{\{a_1a_3\}\{a_2\}\}$, $\Pi_4 = \{\{a_2a_3\}\{a_1\}\}$ and $\Pi_5 = \{\{a_1\}\{a_2\}\{a_3\}\}$.

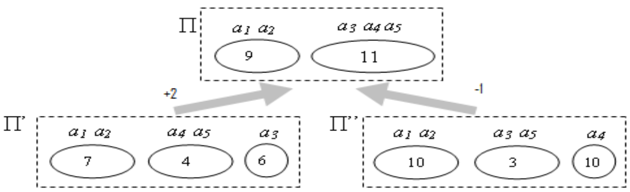
Externalities from Coalition Formation: Traditionally, in game theory, externalities are related to the merger of two coalitions in a system. In more detail, it is a change in value of a given coalition caused by a merge of another two distinct coalitions in the system. More formally, a conventional definition of an externality is as follows:

DEFINITION 1. *Let* Π' *and* Π *be any two coalition structures such that* Π' *contains the disjoint coalitions* C_1, C_2 *and* C_3, *and* Π



$\mathcal{P}(\Pi_1) = [12]$; $\mathcal{P}(\Pi_2) = [18, 10]$;
$\mathcal{P}(\Pi_3) = [12, 8]$; $\mathcal{P}(\Pi_4) = [13, 9]$;
$\mathcal{P}(\Pi_5) = [11.1, 9.9, 15]$

Figure 1: Example of PFG for $A = \{a_1, a_2, a_3\}$



$\mathcal{P}(\Pi) = [9, 11]$; $\mathcal{P}(\Pi') = [7, 4, 6]$; $\mathcal{P}(\Pi'') = [10, 3, 10]$;

Figure 2: Example of two different externalities from formation of coalition $\{a_3a_4a_5\}$ on coalition $\{a_1a_2\}$

is formed from Π' by merging C_2 and C_3. The externality from the formation of the new coalition $C_4 = C_2 \cup C_3$ on coalition C_1 is measured as the value of C_1 in Π minus its value in Π'. That is, $\mathcal{P}(C_1, \Pi) - \mathcal{P}(C_1, \Pi')$.

Now, one characteristic of this conventional definition is that, in general, the value of the externality is dependent upon which two coalitions took part in a merge. This is illustrated in Figure 2, where the merging of $\{a_3\}$ with $\{a_4a_5\}$ induces a different externality on $\{a_1a_2\}$ than the merging of $\{a_4\}$ with $\{a_3a_5\}$. In particular, $\mathcal{P}(\{a_1a_2\}, \Pi) - \mathcal{P}(\{a_1a_2\}, \Pi') = 9 - 7 = 2$, whereas $\mathcal{P}(\{a_1a_2\}, \Pi) - \mathcal{P}'(\{a_1a_2\}, \Pi'') = 9 - 10 = -1$. In other words, the externality is a function of both the embedded coalition (C, Π) and the coalition structure from which Π has been created.

4. NEW REPRESENTATIONS

In this section we present a number of novel representations of games with externalities. These are based on alternative notions of externalities, which are not related to mergers of coalitions. The common element of new representations is that externalities are separated from the elemental values of coalitions.

Total Externalities from Coalition Formation: While the conventional, game-theoretic approach to externalities is pivotal in the analysis of coalition formation processes when viewed as the creation of coalition structures *via mergers of coalitions*, the designer of a multi-agent system is often not concerned with such coalition formation processes *per se*. In other words, the analysis of any given coalition structure does not always need to take into consideration the possible mergers that result in the formation of this structure. For instance, in their seminal paper, Sandholm *et al* [17] argue that there is a much wider scope of activities related to coalition formation than only *how* coalition structures are created. Specifically, they distinguish between: (i) calculating values of every possible coalition; (ii) finding a division of all the agents into exhaustive and disjoint coalitions such that the effectiveness of the system is maximised; and (iii) distributing the payoff among the agents so that the chosen coalition structure is stable. The first issue, considered, for example, in [13], is often completely independent of the coalition formation processes (understood from the game-theoretic perspective of mergers). The other two issues also do not have to be dependent on such processes. In fact, as far as the multi-agent systems literature is concerned, it is often assumed that agents are cooperative [17] [16] [9]. For instance, while solving the coalition structure generation problem, this means that there is no need to consider whether a chosen coalition structure is stable, or how it might be reached, and that is because the participant agents are assumed to accept any outcome.

As a result, in both multi-agent systems and conventional game theory, there has been a need to redefine the concept of externalities in such a way that it does not include all the details of the coalition formation processes when they are irrelevant to an analysis. Consequently, in their derivation of Shapley value for $PFGs$, DeClippel and Serrano [5] propose a concept of an *externality-free value* of C which reflects the performance of this coalition in the absence of externalities from coalition formation.[2]

Intuitively, the externality-free value of coalition C must be calculated in a coalition structure in which C is not subject to any externalities. It is clear that the sole coalition structure which meets this condition is the one containing, apart from C, only singletons.

DEFINITION 2. *The externality-free value of a coalition C, denoted as $v_{ef} : 2^A \to \mathbb{R}$, is the value of the embedded coalition (C, Π) where $\Pi \backslash \{C\}$ is composed only of singletons.*[3] *More formally, $v_{ef}(C) = \mathcal{P}(C, \{C, \{a_j\}_{a_j \in A \backslash C}\})$. We will call $v_{ef}(C)$ an externality-free characteristic function.*

Following [5], every partition function $\mathcal{P}(C; \Pi)$ can be decomposed as follows:[4]

$$\mathcal{P}(C, \Pi) = v_{ef}(C) + \mathcal{T}(C, \Pi); \tag{1}$$

where $\mathcal{T} : \mathcal{C} \to \mathbb{R}$ will be called the *function of total externalities from coalition formation*. This function takes, as input, $(C, \Pi) \in \mathcal{C}$ and outputs a value representing what we call the *total externalities from coalition formation*, which is basically the sum of all the externalities that are induced upon C due to the formation of the other coalitions in Π. We will formally denote this representation as a tuple $PFG_T = \langle A, v_{ef}, \mathcal{T} \rangle$.

EXAMPLE 1. *Given the system introduced in Figure 1, assume that the externality-free characteristic function, as well as the function of total externalities from coalition formation are known. Let $v_{ef}(C)$ be as follows: $v_{ef}(\{a_1\}) = 11.1$, $v_{ef}(\{a_2\}) = 9.9$, $v_{ef}(\{a_3\}) = 15$, $v_{ef}(\{a_1a_2\}) = 18$, $v_{ef}(\{a_1a_3\}) = 12$, $v_{ef}(\{a_2a_3\}) = 13$ and $v_{ef}(\{a_1a_2a_3\}) = 12$. Furthermore, let the total externalities from coalition formation be equal to zero except: $\mathcal{T}(\{a_3\}, \Pi_2) = -5$; $\mathcal{T}(\{a_{2.1}\}, \Pi_3) = -1.9$; and $\mathcal{T}(\{a_1\}, \Pi_4) = -2.1$, where $\Pi_1, \ldots, \Pi_5$ are as defined in Figure 1. It is easy to check that the above functions, i.e., $v_{ef}(C)$ and $\mathcal{T}(C, \Pi)$, together constitute a PFG_T representation of the system in Figure 1.*

Note that the function $\mathcal{T}(C, \Pi)$, in formula (1), has an interesting interpretation. It represents the combined externalities on the value of (C, Π) from any possible coalition formation process that starts with structure $\{C, \{a_j\}_{a_j \in A \backslash C}\}$. This is illustrated in Figure 3. Specifically, starting from Π'''', whichever path we undertake to reach Π the sum of all externalities from coalition formation on $\{a_1a_2\}$ (denoted at the edges on Figure 3) is always equal to $\mathcal{T}(\{a_1a_2\}, \Pi) = 5$. We formulate the above as:

[2] In the multi-agent system literature, the same concept was independently developed in [9].
[3] Note that 2^A denotes the set of all the possible coalitions that can be formed by agents in set A.
[4] As a notational convenience, we drop the parenthesis when referring to an embedded coalition that is an argument of a function. In other words, we write $f(C, \Pi)$ instead of $f((C, \Pi))$.

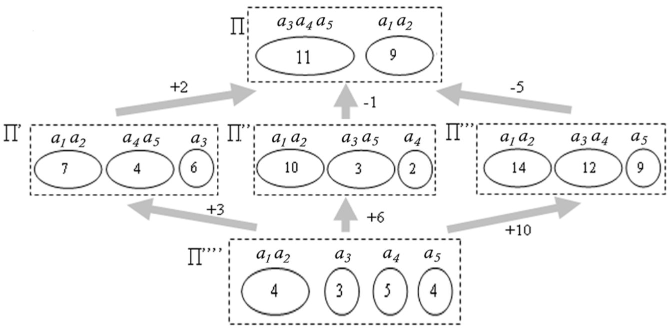


Figure 3: The possible paths leading from coalition structure $\Pi'''' := \{\{a_1a_2\}\ \{a_3\}\{a_4\}\{a_5\}\}$ to $\Pi := \{\{a_1a_2\}\{a_3a_4a_5\}\}$

OBSERVATION 1. *$\mathcal{T}(C,\Pi)$ equals the combined externalities from any possible coalition formation process on C (when such a process is viewed as a number of sequential mergers of two coalitions) starting at structure $\{C, \{a_j\}_{a_j \in A\setminus C}\}$ and finishing at Π.*

In Figure 3 for example, when $\{a_4, a_5\}$ is formed, coalition $\{a_1a_2\}$ is subject to the externality: $\mathcal{P}(\{a_1a_2\},\Pi') - \mathcal{P}(\{a_1a_2\},\Pi'''') = 3$. Moreover, when $\{a_3, a_4, a_5\}$ is formed, $\{a_1a_2\}$ is subject to the following externality: $\mathcal{P}(\{a_1a_2\},\Pi) - \mathcal{P}(\{a_1a_2\},\Pi') = 2$. In total, on this particular path leading from Π'''' to Π, coalition $\{a_1a_2\}$ is subject to the following externalities:

$$\mathcal{P}(\{a_1a_2\},\Pi') - \mathcal{P}(\{a_1a_2\},\Pi'''') + \mathcal{P}(\{a_1a_2\},\Pi) - \mathcal{P}(\{a_1a_2\},\Pi') = \mathcal{P}(\{a_1a_2\},\Pi) - \mathcal{P}(\{a_1a_2\},\Pi'''') = \mathcal{T}(\{a_1a_2\},\Pi),$$

Theoretically, it is possible to analyse issues related to externalities from coalition formation using the conventional formulation in Definition 1. However, this requires considering an exponential number of potential mergers.[5] The advantage of the PFG_T representation is that it allows us to observe and analyse issues related to (total) externalities with much lower complexity.

It should be observed that the notion of total externalities is still based on how the merging of coalitions affects other coalitions in the system. In the next section we will show that one can look at externalities from a different perspective which is independent of these processes.

Contribution vs. Value: Conventional PFG research, as well as the above notion of total externalities from coalition formation, focuses on the value of $\mathcal{P}(C,\Pi)$ (*i.e.*, the value of C in Π.). Our second approach takes a different perspective and looks at the *contribution of C to* Π rather than the *value of C in* Π. Specifically, let us divide this contribution into two parts. The first represents the contribution of C to Π in the absence of any influence on, or from, the other coalitions in Π. This part is, by definition, independent from Π. The second part represents how the functioning of C affects the other coalitions in Π. Consider the following example:

EXAMPLE 2. *Consider the following coalition structure $\Pi' = \{\{a_1a_2\}\{a_3\}\{a_4a_5\}\}$ which is taken from Figure 2, and assume that the coalitions in Π' achieve, in the absence of any influence on each other, the following contributions respectively: 5, 9 and 2. Now, allow for such an influence and assume that $\{a_1a_2\}$ increases the effectiveness of $\{a_4a_5\}$ by 3 and decreases the effectiveness of $\{a_3\}$ by 4. Moreover, suppose that $\{a_3\}$ influences $\{a_4a_5\}$ by -1, and has no influence on $\{a_1a_2\}$. Finally, suppose that $\{a_4a_5\}$ influences $\{a_1a_2\}$ by 2 and $\{a_3\}$ by 1. All these influences are depicted in Figure 4, panel (a). It is straightforward to calculate that the total values of coalitions in Π' are 7, 6 and 4, but their total contribution to Π' are 4, 8 and 5, respectively. Of course, in both cases, the value of the entire structure is the same and is equal to 17.*

[5]Note that the number of possible mergers of two coalitions is greater than the number of possible coalition structures (see [17] for more details).

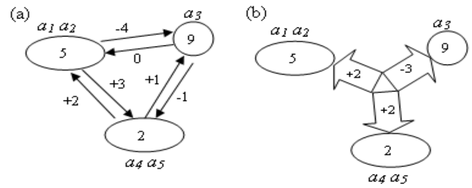
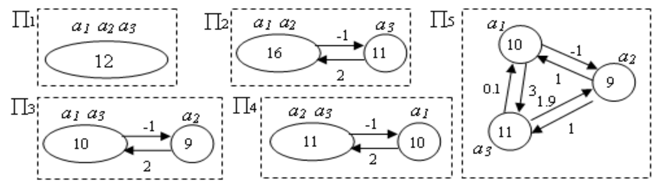


Figure 4: Coalitions affecting each other in coalition structure Π' as defined in Figure 2



$v_r(\{a_1\}) = 10;$	$\overrightarrow{\mathcal{E}}(\{a_3\},\Pi_2) = [2];$
$v_r(\{a_2\}) = 9;$	$\overrightarrow{\mathcal{E}}(\{a_1a_2\},\Pi_2) = [-1]$
$v_r(\{a_3\}) = 11;$	$\overrightarrow{\mathcal{E}}(\{a_2\},\Pi_3) = [2];$
$v_r(\{a_1a_2\}) = 16;$	$\overrightarrow{\mathcal{E}}(\{a_1a_3\},\Pi_3) = [-1];$
$v_r(\{a_1a_3\}) = 10;$	$\overrightarrow{\mathcal{E}}(\{a_1\},\Pi_4) = [2];$
$v_r(\{a_2a_3\}) = 11;$	$\overrightarrow{\mathcal{E}}(\{a_2a_3\},\Pi_4) = [-1];$
$v_r(\{a_1a_2a_3\}) = 12;$	$\overrightarrow{\mathcal{E}}(\{a_1\},\Pi_5) = [-1, 3];$
	$\overrightarrow{\mathcal{E}}(\{a_2\},\Pi_5) = [1, 1];$
	$\overrightarrow{\mathcal{E}}(\{a_3\},\Pi_5) = [0.1, 1.9];$

Figure 5: Example of $PFG_{\overrightarrow{E}}$ for $A = \{a_1, a_2, a_3\}$

Next, instead of looking at how a given coalition $C \in \Pi$ affects the other coalitions in Π, we will look at how C is affected by the other coalitions in Π. Let us consider the following example.

EXAMPLE 3. *Consider $\Pi' = \{\{a_1a_2\}\{a_3\}\{a_4a_5\}\}$ from Figure 2, and assume that the coalitions in Π' achieve, in the absence of any influence on one another, the contributions 5, 9 and 2 respectively. Moreover, assume that, in total, coalition $\{a_1a_2\}$ is affected by 2, while $\{a_3\}$ is affected by -3 and $\{a_4a_5\}$ by 2. These influences are depicted in Figure 4, panel (b).*

As opposed to Example 2, which considers the influence of a given coalition on *each of the other coalitions*, Example 3 considers the total influence induced upon a given coalition by *all the other coalitions* in the coalition structure. We will call the first type of influence *outward operational externalities*, and denote them by $\overrightarrow{E}$. The second type of the influence will be called *inward operational externalities* and be denoted by $\overleftarrow{E}$. Finally, the values of coalitions in the absence of any such operational externalities will be called *residual values*. More formally:

DEFINITION 3. *An outward operational externality $\overrightarrow{E}$ that coalition (C,Π) induces on coalition (C',Π) is an influence of C on C' in Π measured by a change of value of C' caused by the functioning of C. An inward operational externality $\overleftarrow{E}$ that coalition (C,Π) is subject to is the total influence of all the coalitions in $\Pi\setminus\{C\}$ on C measured by a change of value of C caused by the functioning of all coalitions in $\Pi\setminus\{C\}$.*

DEFINITION 4. *A residual value, denoted $v_r(C)$, reflects the performance of coalition C in the absence of any operational externalities. We call $v_r : 2^A \rightarrow \mathbb{R}$ a residual characteristic function.*

Representation Based on Contribution: Against this background, the performance of C in Π can be represented by a function $\mathcal{V}_{\overrightarrow{E}} : \mathcal{C} \rightarrow \mathbb{R}$ which outputs the coalition C's contribution **to** coalition structure Π. Using both the concept of outward operational externalities and the concept of residual values, this function can be decomposed as follows:

$$\mathcal{V}_{\overrightarrow{E}}(C, \Pi) = v_r(C) + \left\| \overrightarrow{\mathcal{E}}(C, \Pi) \right\|, \tag{2}$$

where $\overrightarrow{\mathcal{E}} : \mathcal{C} \rightarrow \mathbb{R}^{|\Pi|-1}$ is the *function of* $\overrightarrow{E}$ which takes, as input, (C, Π) and outputs a $(|\Pi| - 1)$-dimensional vector representing all $\overrightarrow{E}$ induced by C on all the other coalitions in Π. We will denote this (transposed) vector as:

$$\overrightarrow{\mathcal{E}}(C, \Pi) = [\epsilon_{C_1}(C, \Pi), \ldots, \epsilon_{C_m}(C, \Pi)] \tag{3}$$

where $\Pi \backslash \{C\} = \{C_1, \cdots, C_m\}$, and $\epsilon_{C_i \in \Pi \backslash \{C\}}(C, \Pi)$ is the operational externality induced upon C_i by C.

We will call $\mathcal{V}_{\overrightarrow{E}}$ the *partition function with outward operational externalities.* More formally, $PFG_{\overrightarrow{E}} := \left\langle A, v_r, \overrightarrow{\mathcal{E}} \right\rangle$. Figure 5 gives an example of this representation for the system from Figure 1.

The concept of a representation based on the contribution and not on the value can also be extended to characteristic function games. Specifically, we define the *contribution characteristic function* $v_c : 2^A \rightarrow \mathbb{R}$ that returns, for every coalition C, its contribution to the system. We call the resulting representation $\langle A, v_c \rangle$ the *contribution characteristic function game* representation. The usefulness of this representation will be apparent in Section 5.

Representation Based on Value: Now consider inward operational externalities and the standard PFG representation. As discussed previously, the partition function $\mathcal{P} : \mathcal{C} \rightarrow \mathbb{R}$ outputs the value of coalition C **in** coalition structure Π. Using both the concept of inward operational externalities and the concept of residual values, $\mathcal{P}$ can be decomposed as follows:

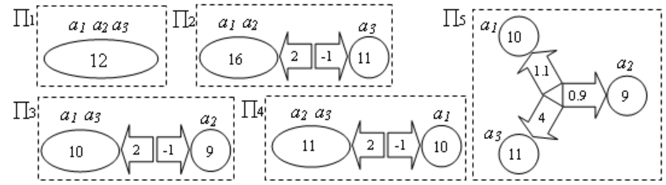
$$\mathcal{P}(C, \Pi) = v_r(C) + \overleftarrow{\mathcal{E}}(C, \Pi), \tag{4}$$

where $\overleftarrow{\mathcal{E}} : \mathcal{C} \rightarrow \mathbb{R}$ is the *function of* $\overleftarrow{E}$ which takes, as input, (C, Π) and outputs a value of $\overleftarrow{E}$ induced upon C by all the other coalitions in Π. Formally, $PFG_{\overleftarrow{E}} := \left\langle A, v_r, \overleftarrow{\mathcal{E}} \right\rangle$. Figure 6 gives an example of this representation for the system from Figure 1.

Representation with Minimal Information: Here, we formalise yet another representation that becomes useful when there is much less information available about the system. Specifically, in this *coalition structure function game* ($CSFG$) representation, the *coalition structure function* $\mathcal{F} : \mathcal{CS} \rightarrow \mathbb{R}$ takes, as input, a coalition structure $\Pi \in \mathcal{CS}$ and outputs its value. Formally, $CSFG := \langle A, \mathcal{F} \rangle$.

In contrast to all the previous representations (*i.e.*, PFG, PFG_T, $PFG_{\overrightarrow{E}}$ or $PFG_{\overleftarrow{E}}$) $CSFG$ can be used to represent systems in which it is not possible to measure the values of embedded coalitions but only the total value of a given coalition structure. Such an assumption about the information structure of the system was made by Sandholm *et al* [17] while considering the coalition structure generation problem.

EXAMPLE 4. *For the 3-agent system from Figure 1 the values of coalition structure function are:* $\mathcal{F}(\Pi_1) = 12$; $\mathcal{F}(\Pi_2) = 28$; $\mathcal{F}(\Pi_3) = 20$; $\mathcal{F}(\Pi_4) = 22$; *and* $\mathcal{F}(\Pi_5) = 36$.



$v_r(\{a_1\}) = 10$; $\overleftarrow{\mathcal{E}}(\{a_3\}, \Pi_2) = -1$;
$v_r(\{a_2\}) = 9$; $\overleftarrow{\mathcal{E}}(\{a_1 a_2\}, \Pi_2) = 2$;
$v_r(\{a_3\}) = 11$; $\overleftarrow{\mathcal{E}}(\{a_2\}, \Pi_3) = -1$;
$v_r(\{a_1 a_2\}) = 16$; $\overleftarrow{\mathcal{E}}(\{a_1 a_3\}, \Pi_3) = 2$;
$v_r(\{a_1 a_3\}) = 10$; $\overleftarrow{\mathcal{E}}(\{a_1\}, \Pi_4) = -1$;
$v_r(\{a_2 a_3\}) = 11$; $\overleftarrow{\mathcal{E}}(\{a_2 a_3\}, \Pi_4) = 2$;
$v_r(\{a_1 a_2 a_3\}) = 12$; $\overleftarrow{\mathcal{E}}(\{a_1\}, \Pi_5) = 1.1$;
$\overleftarrow{\mathcal{E}}(\{a_2\}, \Pi_5) = 0.9$;
$\overleftarrow{\mathcal{E}}(\{a_3\}, \Pi_5) = 4$;

Figure 6: Example of $PFG_{\overleftarrow{E}}$ for $A = \{a_1, a_2, a_3\}$

5. CLASSES OF GAMES

As mentioned before, a common characteristic of the representations introduced in the previous section is that externalities are separated from elemental (externality-free/residual) values of coalitions. In this section, we focus on the externality part of this decomposition and define classes of games in which externalities meet particular patterns.

Classes: We focus on games with regularities in externalities:

$S_1^{T/\overrightarrow{E}/\overleftarrow{E}}$ In every coalition structure to which they belong, coalitions of the same size are subject to the same total externalities (T) / induce the same sum of outward operational externalities ($\overrightarrow{E}$) / are subject to the same inward operational externalities ($\overleftarrow{E}$).

$S_2^{T/\overrightarrow{E}/\overleftarrow{E}}$ In every coalition structure to which they belong, a given coalition C is subject to the same total externalities (T) / induces the same sum of outward operational externalities ($\overrightarrow{E}$) / is subject to the same inward operational externalities ($\overleftarrow{E}$).

$S_3^{T/\overrightarrow{E}/\overleftarrow{E}}$ In every coalition structure in a given subspace $\mathcal{S}_{i_1, i_2, \ldots, i_k}$, coalitions of the same size are subject to the same total externalities (T) / induce the same sum of outward operational externalities ($\overrightarrow{E}$) / are subject to the same inward operational externalities ($\overleftarrow{E}$).

As an illustrative example, let us consider a cooperative environment in which every coalition of agents creates (or saves) some additional resource. For instance, cooperating agents consume less resource than when acting in smaller groups. Furthermore, this saved amount of the resource is distributed among other coalitions in the system. Although this distribution might depend on the coalitional arrangements of other agents, the amount to distribute is always the same. Every such system belongs to class $S_2^{\overrightarrow{E}}$. Analogous examples can be constructed for other classes.

It can be observed that $S_1^i \subset S_2^i$ and $S_1^i \subset S_3^i$ but not necessarily $S_2^i \subset S_3^i$ where $i = T, \overrightarrow{E}/\overleftarrow{E}$. For example, the outward operational externalities in Figure 5 meet the characteristics of class $S_1^{\overrightarrow{E}}$. In particular, all coalitions of size 1 induce outward operational externalities which always sum to 2 and coalitions of size 2 induce externalities that always sum to -1. Since the three-agent system considered in this paper belongs to class $S_1^{\overrightarrow{E}}$, then it also belongs to classes $S_3^{\overrightarrow{E}}$. It is easy to check that this particular system also belongs to $S_2^{\overrightarrow{E}}$.

Reduction of Classes $S_{1/2}^{\overrightarrow{E}/\overleftarrow{E}/T}$ to Contribution Characteristic Function: Let us consider a certain embedded coalition (C,Π) in a system belonging to class $S_2^{\overrightarrow{E}}$ and modeled using the $PFG_{\overrightarrow{E}}$ representation. As stated in the definition of this class, the sums of outward operational externalities $\left\|\overrightarrow{\mathcal{E}}(C,\Pi)\right\|$ are not functions of an embedded coalition (C,Π) but only of coalition C. Consequently, formula (2) can be reduced as follows:

$$\mathcal{V}_{\overrightarrow{E}}(C,\Pi) = v_r(C) + \left\|\overrightarrow{\mathcal{E}}(C)\right\| =: v_{\overrightarrow{E}}(C), \tag{5}$$

where $v_{\overrightarrow{E}}(C)$ encompasses both the value of the residual characteristic function and the value of outward operational externalities for coalition C. In other words, $v_{\overrightarrow{E}}$ is the contribution characteristic function (see Section 4).

EXAMPLE 5. *As observed, the system in Figure 5 belongs to class $S_2^{\overrightarrow{E}}$. The $PFG_{\overrightarrow{E}}$ representation of this game can be reduced to the corresponding contribution characteristic function game representation. Specifically, using formula (5) we obtain the following values:* $v_{\overrightarrow{E}}(\{a_1a_2a_3\}) = 12; v_{\overrightarrow{E}}(\{a_1a_2\}) = 15; v_{\overrightarrow{E}}(\{a_1a_3\}) = 9; v_{\overrightarrow{E}}(\{a_2a_3\}) = 10; v_{\overrightarrow{E}}(\{a_1\}) = 12; v_{\overrightarrow{E}}(\{a_2\}) = 11;$ *and* $v_{\overrightarrow{E}}(\{a_3\}) = 13$.

It should be stressed that this reduction can be used if we are interested in contributions of various coalitions and not in the strategic choices of agents as $v_{\overrightarrow{E}}$ does not contain information about actual payoffs of coalitions. Similar reasoning holds for any system belonging to either class $S_{1/2}^{T}$ or $S_{1/2}^{\overleftarrow{E}}$. However, the difference here is that the reduction of both PFG_T and $PFG_{\overleftarrow{E}}$ simply results in the conventional characteristic function game.[6]

Reduction for Classes $S_3^{\overrightarrow{E}/\overleftarrow{E}/T}$: Let us consider a system of class $S_3^{\overrightarrow{E}}$ represented with $PFG_{\overrightarrow{E}}$. In this class, the sum of outward operational externalities induced by an embedded coalition (C,Π) are a function of both $|C|$ and $\mathcal{S}_{i_1,\dots,i_k} \ni \Pi$. Therefore, without loss of generality, $\overrightarrow{\mathcal{E}}$ can be redefined as a function of both $|C|$ and $\mathcal{S}_{i_1,\dots,i_k}$. Consequently, formula (2) yields:

$$\mathcal{V}_{\overrightarrow{E}}(C,\Pi) = v_r(C) + \left\|\overrightarrow{\mathcal{E}}(|C|,\mathcal{S}_{i_1,\dots,i_k})\right\|. \tag{6}$$

Now, let $\overrightarrow{\xi}(\Pi) = [\left\|\overrightarrow{\mathcal{E}}(i_1,\mathcal{S}_{i_1,\dots,i_k})\right\|,\dots,\left\|\overrightarrow{\mathcal{E}}(i_k,\mathcal{S}_{i_1,\dots,i_k})\right\|]$ denote a function that assigns to every coalition structure $\Pi \in \mathcal{S}_{i_1,\dots,i_k}$ a transposed vector of sums of outward operational externalities. Since the sum of the outward operational externalities in Π is the same for every $\Pi \in \mathcal{S}_{i_1,\dots,i_k}$, then $\overrightarrow{\xi}$ can be redefined as a function of $\mathcal{S}_{i_1,\dots,i_k}$, which we denote $\overrightarrow{\xi}^*(\mathcal{S}_{i_1,\dots,i_k})$. In this way, the $PFG_{\overrightarrow{E}}$ representation of a system from class $S_3^{\overrightarrow{E}}$ can be reduced to a tuple $\left\langle A, v_r, \overrightarrow{\xi}^*\right\rangle$.

Our approach here is conceptually similar to that of Ieong and Shoham [7]. Their representation is based on patterns with assigned values, or *pattern→value*, where *pattern* is a logical expression with agents as atoms. If agents in a coalition satisfy a *pattern* the value of a coalition is increased accordingly (by *value*). In our aproach a *pattern* is a subspace $\mathcal{S}_{i_1,\dots,i_k}$ and a *value* is a vector of sums of outward operational externalities. The advantage of redefining $\overrightarrow{\xi}(\Pi)$ as $\overrightarrow{\xi}^*(\mathcal{S}_{i_1,\dots,i_k})$ is that the latter function requires substantially less memory (a number of all integer partitions of n and not a number of all coalition structures).

[6]This is because these representations are based on the values of the coalitions and not their contributions to the system.

EXAMPLE 6. *Again consider the system from Figure 5. There are altogether 3 coalition structures in subspace $\mathcal{S}_{2,1}$, namely Π_2, Π_3 and Π_4. From Figure 5 we find that $\overrightarrow{\xi}(\Pi_2) = [-1,2]$; $\overrightarrow{\xi}(\Pi_3) = [-1,2]$; and $\overrightarrow{\xi}(\Pi_3) = [-1,2]$. Therefore, $\overrightarrow{\xi}(\mathcal{S}_{2,1}) = [-1,2]$.*

Again, it should be stressed that that this reduction can be used if we are interested in contributions of various coalitions and not in the strategic choices of agents. This is because the reduced representation $\left\langle A, v_r, \overrightarrow{\xi}^*\right\rangle$ does not contain information about actual payoffs of embedded coalitions.

Similar reasoning holds for classes $S_3^{\overleftarrow{E}}$ and S_3^{T}, *i.e.*, PFG_T and $PFG_{\overleftarrow{E}}$ can be reduced to $\left\langle A, v_r, \overleftarrow{\xi}^*\right\rangle$ and $\left\langle A, v_{ef}, \xi^{T*}\right\rangle$ respectively, where function $\overleftarrow{\xi}^*$ and ξ^{T*} are defined similarly to $\overrightarrow{\xi}^*$. However, in contrast to $\left\langle A, v_r, \overrightarrow{\xi}^*\right\rangle$, reduced representations $\left\langle A, v_r, \overleftarrow{\xi}^*\right\rangle$ and $\left\langle A, v_{ef}, \xi^{T*}\right\rangle$ contain information from which payoffs of embedded coalitions can be computed.

6. INFORMATION, EXPRESSIVENESS AND CONCISENESS

In this section, we assess the extent of information needed in order to model a particular system with each of the representations. Furthermore, referring to the criteria outlined in the introduction, we will discuss the expressivity and conciseness of all the representations.

6.1 Information Requirements

In terms of information about the system, PFG_T requires exactly the same information as PFG. In other words, for every system represented with PFG_T, it is possible to derive the corresponding PFG representation and *vice versa*, *i.e.*:

$$PFG_T \iff PFG. \tag{7}$$

In order to derive PFG from PFG_T, formula (1) should be applied. The derivation of PFG_T from PFG is a little more complicated. First, following Definition 2, the externality-free characteristic function should be obtained from particular coalition structures. Second, the total externalities from coalition formation should be computed by applying formula (1) to all embedded coalitions in every coalition structure.

Let us now consider the representations that are based on operational externalities. Following Definition 3, inward operational externalities induced upon (C,Π) are a measure of the combined influence of all the other coalitions in Π upon C. This influence can be extracted from the transposed vectors of outward operational externalities from formula (3). In particular, for every embedded coalition:

$$\overleftarrow{\mathcal{E}}(C,\Pi) = \sum_{\forall C' \in \Pi \setminus \{C\}} \epsilon_C(C',\Pi). \tag{8}$$

For instance, this identity can be used to derive inward operational externalities in panel (b) of Figure 4 from outward operational externalities in panel (a) of the figure. Now, by adding $v_r(C)$ to both sides of (8), we obtain a formula that links $PFG_{\overleftarrow{E}}$ to $PFG_{\overrightarrow{E}}$. In particular, for every embedded coalition:

$$v_r(C) + \overleftarrow{\mathcal{E}}(C,\Pi) = v_r(C) + \sum_{\forall C' \in \Pi \setminus \{C\}} \epsilon_C(C',\Pi). \tag{9}$$

Note that formula (9) allows us to derive $PFG_{\overleftarrow{E}}$ from $PFG_{\overrightarrow{E}}$ but not *vice versa, i.e.*:

$$PFG_{\overrightarrow{E}} \Rightarrow PFG_{\overleftarrow{E}} \text{ but } PFG_{\overrightarrow{E}} \not\Leftarrow PFG_{\overleftarrow{E}} \tag{10}$$

Furthermore, let us consider both PFG and $PFG_{\overleftarrow{E}}$. Formula (4) directly links the two representations. Specifically, it shows that it is possible to derive PFG from $PFG_{\overleftarrow{E}}$. Obviously, the opposite is not possible, *i.e.*:

$$PFG_{\overleftarrow{E}} \Rightarrow PFG \text{ but } PFG_{\overleftarrow{E}} \not\Leftarrow PFG \tag{11}$$

In this context, it should be stressed that an externality-free characteristic function is not, in general, identical to a residual characteristic function. In more detail, although both Definitions 4 and 2 characterize the value of C in the absence of externalities, the meanings of $v_r(C)$ and $v_{ef}(C)$ are different. This is because the definitions of externalities under both representations are distinct. Specifically, while inward operational externalities represent the combined value of the system's influence on a given coalition, externalities from coalition formation represent changes in this coalition's value caused by coalition formation processes (*i.e.*, merging of coalitions). For instance, Example 1 and Figure 5 contain representations of exactly the same system but, evidently, $v_{ef}(C) \neq v_r(C)$ (*e.g.*, $11.1 = v_{ef}(\{a_1\}) \neq v_r(\{a_1\}) = 10$). What causes this difference? From Definition 2, $v_{ef}(\{a_1\})$ is calculated as the value of $\{a_1\}$ embedded in $\Pi_5 = \{\{a_1\}\{a_2\}\{a_3\}\}$ (since there are no externalities in this coalition structure). This does not mean that agents do not influence each other in Π_5 but that there are no externalities stemming from processes of coalition formation. The fact that agents affect each others' performance is visible from outward operational externalities under $PFG_{\overrightarrow{E}}$. In particular, the performance of $\{a_1\}$ is influenced by $\{a_2\}$ with 1 and by $\{a_3\}$ with 0.1. Thus, the functioning of $\{a_2\}$ and $\{a_3\}$ enhance the performance of $\{a_1\}$ in Π_5 from its residual value of 10 to its externality-free value of 11.1.

For $CSFG$, the following holds:

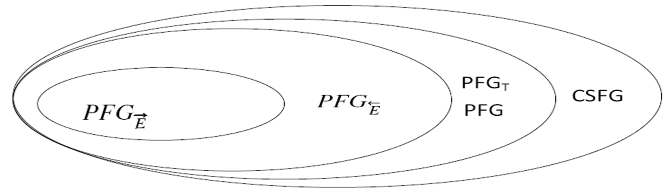
$$\mathcal{F}(\Pi) = \sum_{\forall(C,\Pi)} \mathcal{P}(C,\Pi) = \sum_{\forall(C,\Pi)} \mathcal{V}_{\overrightarrow{E}}(C,\Pi) \tag{12}$$

Whereas the first part of this identity directly stems from the definition of $CSFG$, the intuition behind the second part is more involved. Under a PFG representation, the partition function $\mathcal{P}(C,\Pi)$ denotes the value of C **in** Π, whereas the function $\mathcal{V}_{\overrightarrow{E}}(C,\Pi)$ denotes the value that C contributes **to** Π. Although, in general, $\mathcal{P}(C,\Pi)$ does not have to be equal to $\mathcal{V}_{\overrightarrow{E}}(C,\Pi)$, the second part of identity (12) must hold as both representations encompass all the elements that constitute the value of Π. For example, see how, in Figure 1, we have: $\mathcal{P}(\{a_1\},\Pi_4) + \mathcal{P}(\{a_2,a_3\},\Pi_4) = 9 + 13 = 22$, and in Figure 5 we have: $\mathcal{V}_{\overrightarrow{E}}(\{a_1\},\Pi_4) + \mathcal{V}_{\overrightarrow{E}}(\{a_2,a_3\},\Pi_4) = (10-1) + (11+2) = 22$.

The set of relationships in Figure 7 summarizes the above discussion. Intuitively, the $PFG_{\overrightarrow{E}}$ representation requires the most information about the system, whereas $CSFG$ requires the least. In more detail, if the residual characteristic function and the values of outward operational externalities are known, the system can be modeled using $PFG_{\overrightarrow{E}}$. However, if inward instead of outward operational externalities are available, the system can be modeled using $PFG_{\overleftarrow{E}}$ but not $PFG_{\overrightarrow{E}}$. Moreover, if the externality-free characteristic function as well as the total externalities from coalition formation ($\mathcal{T}(C,\Pi)$) are known one can use PFG_T. On the other hand, if the partition function is known one can use PFG. Naturally, as mentioned before, both PFG_T and PFG can be used interchangeably. Finally, if only the values of coalition structures are known then $CSFG$ should be used.

6.2 Expressiveness and Conciseness

First, let us focus on the criterion of expressivness. We consider a representation to be fully expressive if it can represent any game.



$$\underset{\langle A, v_r, \overrightarrow{\mathcal{E}}\rangle}{PFG_{\overrightarrow{E}}} \Longrightarrow \underset{\langle A, v_r, \overleftarrow{\mathcal{E}}\rangle}{PFG_{\overleftarrow{E}}} \Longrightarrow (\underset{\langle A, v_{ef}, \mathcal{I}\rangle}{PFG_T} \Longleftrightarrow \underset{\langle A, \mathcal{P}\rangle}{PFG}) \Longrightarrow \underset{\langle A, \mathcal{F}\rangle}{CSFG}$$

$\xleftarrow{\text{more information}}$ $\xrightarrow{\text{less information}}$

Figure 5: Relationship between different representations

PROPOSITION 1. *The PFG_T, $PFG_{\overrightarrow{E}}$, and $PFG_{\overleftarrow{E}}$ representations are fully expressive.*

PROOF. In the spirit of [7], we prove expressivity by showing that any partition function game can be expressed as either PFG_T, $PFG_{\overrightarrow{E}}$ or $PFG_{\overleftarrow{E}}$. Let functions $\mathcal{P}(C,\Pi)$, $v_{ef}(C)$, $\mathcal{T}(C,\Pi)$, $v_r(C)$, $\overrightarrow{\mathcal{E}}(C,\Pi)$ and $\overleftarrow{\mathcal{E}}(C,\Pi)$ be defined for an arbitrary system. Relationship (7) shows that, for every PFG representation, there exists a unique corresponding PFG_T representation. Furthermore, one can always derive $\mathcal{P}(C,\Pi)$ from $v_r(C)$, $\overrightarrow{\mathcal{E}}(C,\Pi)$ and $\overleftarrow{\mathcal{E}}(C,\Pi)$ as established by relationships (11) and (10). Since the system is arbitrary, the proposition holds. $\square$

Now, we will analyse and compare the conciseness of all representations. Let us consider an arbitrary game with externalities.

- $CSFG$: The *coalition structure function ($\mathcal{F}$)* returns a value for every coalition structure. This makes the total number of values equal to:

$$\sum_{s=1}^{n} S(n,s) \tag{13}$$

  where $S(n,s)$, the number of coalition structures containing s coalitions, is calculated as follows [4]:

$$S(n,s) = \frac{1}{s!}\sum_{i=0}^{s-1}(-1)^i \binom{s}{i}(s-i)^n \tag{14}$$

- PFG: For every possible embedded coalition, the value of the *partition function ($\mathcal{P}$)* needs to be defined. This implies that, for every coalition structure CS, we need to define $|CS|$ values. Therefore, the total number of values is:

$$\sum_{s=1}^{n} S(n,s) \times s \tag{15}$$

- PFG_T: For every possible coalition, this representation requires storing an *externality-free value (v_{ef})*. Moreover, for every embedded coalition $(C,CS) : C \in CS, |CS| < n$, it requires storing the *function of total externalities from coalition formation ($\mathcal{T}$)* . As a result, PFG_T requires storing the following number of values:

$$(2^n - 1) + \sum_{s=1}^{n-1} S(n,s) \times s \tag{16}$$

- $PFG_{\overleftarrow{E}}$: This representation requires storing a *residual value (v_r)* for all the possible coalitions. It also requires storing

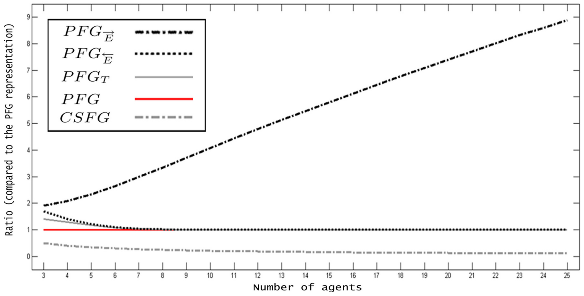


Figure 8: A comparison of the memory requirements of different representations given different numbers of agents.

the *inward operational externality* ($\overleftarrow{\mathcal{E}}$) for every possible embedded coalition $(C, CS) : C \in CS$. This makes the total number of values to be stored:

$$(2^n - 1) + \sum_{s=1}^{n} S(n, s) \times s \tag{17}$$

- $PFG_{\overrightarrow{E}}$: For every possible coalition, $PFG_{\overrightarrow{E}}$ needs to store a *residual value* v_r. Moreover, for every possible embedded coalition $(C, CS) : C \in CS$, it needs to store a vector of $(|CS| - 1)$ values representing the *outward operational externalities* ($\overrightarrow{\mathcal{E}}$). The total number of values is, then:

$$(2^n - 1) + \sum_{s=1}^{n} S(n, s) \times s \times (s - 1) \tag{18}$$

Figure 8 compares the memory requirements of different representations, with conventional PFG being the benchmark. As can be seen from the figure, PFG_T and $PFG_{\overleftarrow{E}}$ require almost the same memory as PFG. Moreover, the figure clearly shows that, although $PFG_{\overrightarrow{E}}$ requires more memory, the increase is linear in the number of agents (*e.g.*, given 25 agents, the memory requirements of $PFG_{\overrightarrow{E}}$ are 9 times as much as those of PFG). Formally, the following proposition holds:

PROPOSITION 2. *The PFG_T and $PFG_{\overleftarrow{E}}$ representations of any game with externalities take only marginally more space than the PFG representation of the same game. $PFG_{\overrightarrow{E}}$, on the other hand, takes a linear factor (in the number of agents) more space than PFG representation.*

Conversely, for certain games our new representations require exponentially less space than the conventional PFG representation.

PROPOSITION 3. *For certain games PFG_T, $PFG_{\overrightarrow{E}}$ and $PFG_{\overleftarrow{E}}$ can be reduced to representations that are exponentially more concise than PFG.*

We will demonstrate this for the classes from Section 5. As for $PFG_{\overrightarrow{E}}$, games belonging to class $S_1^{\overrightarrow{E}}$ (or $S_2^{\overrightarrow{E}}$) can be reduced to the contribution characteristic function. Such a reduction would not be possible if the same system was represented with PFG. This can be seen by comparing Figures 1 and 5. Specifically, while no pattern of externalities is visible under the PFG representation in Figure 1 (as externalities are included in coalition values), this pattern becomes visible under the $PFG_{\overleftarrow{E}}$ representation in Figure 5 (see how the sum of externalities induced by coalitions of the same size is always the same). As for PFG_T and $PFG_{\overleftarrow{E}}$, examples of more concise reduced representations (*i.e.*, $\left\langle A, v_r, \overleftarrow{\xi}^* \right\rangle$ and $\left\langle A, v_{ef}, \xi^{T*} \right\rangle$) are provided in Section 5 for games belonging to classes $S_3^{T/\overrightarrow{E}/\overleftarrow{E}}$.

7. COALITION STRUCTURE GENERATION

In this section we evaluate the efficiency of the new representations by proposing a number of approaches to the coalition structure generation problem for games with externalities. This section is organized as follows. First, we analyse the existing CSG algorithms from the point of view of available information about the system (Subsection 7.1). Second, we consider algorithms for classes of games with externalities introduced in Section 5. In particular, we present an extension of Rahwan *et al*'s anytime algorithm [16] (Subsection 7.2). Finally, we demonstrate that the approach proposed in Michalak *et al* [9] to solve the coalition structure generation problem in four particular systems with externalities, can, in fact, be used to solve any multi-agent system with externalities for which sufficient information is available. PFG_T, $PFG_{\overrightarrow{E}}$ or $PFG_{\overleftarrow{E}}$ constitute a natural and convenient representations for this approach (Subsection 7.3).

7.1 Approaches to the CSG Problem

The literature on coalition structure generation has mainly focused on systems that can be represented using $CFGs$. A short but informative review of the main approaches can be found in [14]. Basically, CSG algorithms for characteristic function games can be divided into three general categories: (a) dynamic programming algorithms that generate optimal solution(s) (*e.g.*, [15, 19]); (b) heuristics which provide no guarantee on solution quality (*e.g.*, [18]); and (c) anytime optimal algorithms (*e.g.*, [16]). The current state-of-the-art algorithm of Rahwan and Jennings [14] combines dynamic programming techniques from [15] with the anytime approach in [16].

All the above approaches are, in general, inappropriate if the system is characterised by non-negligible externalities. It is easy to show that, in the general case, finding an optimal coalition structure requires searching the whole space of possible solutions. This is because, even if all the coalition structures but one have been evaluated, this one remaining coalition structure may deliver a significantly better performance. The situation changes if some additional information is known about the system. For example, Michalak *et al* showed that, if the externalities are known to be either only positive or only negative, then it is possible to bound the coalition values and to represent the system in the form of an approximated characteristic function [9]. Building upon this observation, they showed that the anytime algorithm of Rahwan et al [16] can be refined and applied to solve the coalition structure generation problem in systems with these types of externalities.

In Section 6 we argued that the available information is one of the key factors that can influence a system designer's choice of representation. In this context, let us divide the information about the system into three categories: (A) information that is known *ex ante* and can be used as an input to the algorithm; (B) information that can be obtained after the algorithm has commenced (*ex post*); and (C) unobtainable information. In the CFG literature it is usually assumed that the non-existence of externalities, as well as the characteristic function, both belong to category A, *i.e.*, they constitute the algorithm's input (*e.g.*, [16]). In contrast, Sandholm *et al* [17] considered, among other cases, the case where the knowledge about the non-existence of externalities was available *ex ante*, but the characteristic function itself was not; it belonged to category C. To be more precise, only the values of coalition structures were available *ex post*. In other words, the coalition structure generation problem in [17] is an optimization problem under the $CSFG$ representation with the additional knowledge that there are no externalities in the system. Michalak *et al*, on the other hand, considered cases where the nature of the domain was known *ex-ante*, including whether externalities are positive or negative and whether coalition values meet the condition of super- or sub-additivity [9]. However, it was assumed that the value of the partition function itself was information of category B, *i.e.*, it could be derived from the system if needed.

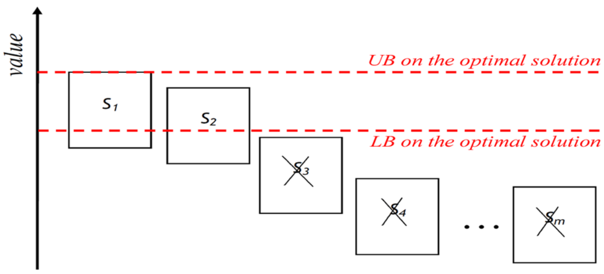


Figure 9: Pruning in Rahwan et al's CSG algorithm. Subspaces are indexed from the most to the least promising

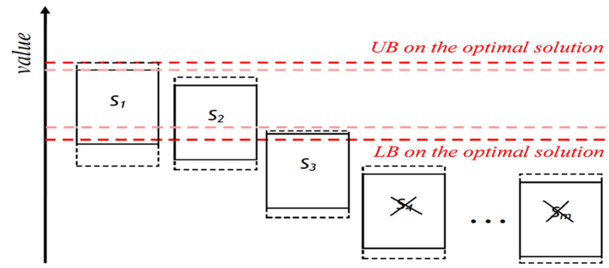


Figure 10: Pruning in games with externalities. Subspace and system bounds are now less tight compared to Figure 9

In the next subsections we will propose a number of solutions to the coalition structure generation problem under the new representations. For clarity of the exposition, we assume that the information needed to represent the system using different representations is known *ex ante*. However, the proposed solutions hold also when only the class of the system is known *ex ante*, whereas the values of relevant function have to be obtained *ex post*.

7.2 Games with Patterns

In this subsection, we consider how to solve the coalition structure generation problem for games with externalities that belong to the classes defined in Section 5.

Classes $S_1^{T/\overrightarrow{E}/\overleftarrow{E}}$ and $S_2^{T/\overrightarrow{E}/\overleftarrow{E}}$: As argued in Section 5, it is possible to reduce the $PFG_{T/\overleftarrow{E}}$ representations for systems belonging to classes $S_{1/2}^{T/\overleftarrow{E}}$ to the CFG representations. This means that, with the characteristic function as an input, the coalition structure generation problem for this system can be solved using any of the available algorithms in the CFG literature.

Importantly, the same reasoning holds for $PFG_{\overrightarrow{E}}$. As shown in Section 5, this representation can be reduced to the corresponding contribution characteristic function game representation. Since there is no difference between the *combined contribution* of all the embedded coalitions to the coalition structure and the *combined value* of these coalitions (see equation (12)), then both representations deliver exactly the same optimal solution. Thus, the contribution characteristic function $v_{\overrightarrow{E}}$ can be used as an input to any of the available algorithms in the CFG literature and deliver the correct outcome for the game with externalities belonging to class $S_{1/2}^{\overrightarrow{E}}$.

Class $S_3^{T/\overrightarrow{E}/\overleftarrow{E}}$: Rahwan *et al*'s anytime coalition structure generation algorithm takes advantage of the fact that there are no externalities in the system. This means that the value of any coalition always remains unchanged and so it is possible to evaluate properties of coalitions from the characteristic function and determine which subspaces are promising and should be searched. During the search, a progressively better bound from the optimal is established until an optimal solution is found.

The most important elements of Rahwan *et al*'s approach, relevant to our refinement, are as follows. Firstly, the space of all coalitions is divided into ordered lists of the same size j (denoted L_j). The algorithm calculates the maximal and average values for every such list (denoted max_j and avg_j, respectively). Using this information, it is possible to generate, for every subspace $\mathcal{S}_{i_1,i_2,\ldots,i_k}$, the upper bound $UB(\mathcal{S}_{i_1,i_2,\ldots,i_k}) := \sum_{j=i_1,\ldots,i_k} max_j$ as well as the average value $AVG(\mathcal{S}_{i_1,i_2,\ldots,i_k}) := \sum_{j=i_1,\ldots,i_k} avg_j$. The subspace with the highest upper bound determines the upper bound of the entire system, while the system's lower bound is shown to be the highest average from all $AVG(\mathcal{S}_{i_1,i_2,\ldots,i_k})$. Any subspace with an upper bound lower than the system's lower bound is pruned from the search space (this is illustrated in Figure 9). After that, the most promising subspace is searched, and then the second most promising, and so on, until a structure is found that has a value greater than the upper bound of all remaining subspaces. The above approach was shown to run efficiently for the most popular value distributions.

In the above context, let us consider a system of class $S_3^{\overrightarrow{E}}$ represented with $PFG_{\overrightarrow{E}}$. In this class, sums of outward operational externalities induced by all coalitions (C,Π) of the same size are a function of a subspace $\mathcal{S}_{i_1,\ldots,i_k} \ni \Pi$. This is formally written as $\overrightarrow{\mathcal{E}}_{sum}(|C|, \mathcal{S}_{i_1,\ldots,i_k})$. Consequently, formula (2) yields:

$$\mathcal{V}_{\overrightarrow{E}}(C,\Pi) = v_r(C) + \left\|\overrightarrow{\mathcal{E}}(C,\Pi)\right\| = v_r(C) + \overrightarrow{\mathcal{E}}_{sum}(|C|, \mathcal{S}_{i_1,\ldots,i_k}). \tag{19}$$

Now, let $\overrightarrow{\xi}(\Pi) = [\overrightarrow{\mathcal{E}}_{sum}(i_1, \mathcal{S}_{i_1,\ldots,i_k}), \ldots, \overrightarrow{\mathcal{E}}_{sum}(i_k, \mathcal{S}_{i_1,\ldots,i_k})]$ denote a transposed vector of sums of outward operational externalities induced by all coalitions in $\Pi \in \mathcal{S}_{i_1,\ldots,i_k}$. Since the sum of the outward operational externalities in Π are the same for every $\Pi \in \mathcal{S}_{i_1,\ldots,i_k}$, then $\overrightarrow{\xi}(\Pi)$ can be written as a function of $\mathcal{S}_{i_1,\ldots,i_k}$, *i.e.*, $\overrightarrow{\xi}(\mathcal{S}_{i_1,\ldots,i_k})$.

EXAMPLE 7. *Again consider the system of class $S_1^{\overrightarrow{E}} \subset S_2^{\overrightarrow{E}} \subset S_3^{\overrightarrow{E}}$ from Figure 5. There are altogether 3 coalition structures in subspace $\mathcal{S}_{2,1}$, namely Π_2, Π_3 and Π_4. From Figure 5 we find that $\overrightarrow{\xi}(\Pi_2) = [-1,2]$; $\overrightarrow{\xi}(\Pi_3) = [-1,2]$; and $\overrightarrow{\xi}(\Pi_3) = [-1,2]$. Therefore, $\overrightarrow{\xi}(\mathcal{S}_{2,1}) = [-1,2]$.*

The above analysis leads to the following refinements of Rahwan *et al.*'s algorithm. Firstly, instead of a characteristic function, both the residual characteristic function ($v_r(C)$) and $\overrightarrow{\xi}(\mathcal{S}_{i_1,\ldots,i_k})$ constitute the input. Consequently, the lists L_j ($j = 1, \ldots, |A|$) are constructed from $v_r(C)$. Secondly, we have:

$$UB(\mathcal{S}_{i_1,\ldots,i_k}) = \left\|\overrightarrow{\xi}(\mathcal{S}_{i_1,\ldots,i_k})\right\| + \sum_{j=i_1,\ldots,i_k} max_j \text{ and}$$

$$AVG(\mathcal{S}_{i_1,\ldots,i_k}) = \left\|\overrightarrow{\xi}(\mathcal{S}_{i_1,\ldots,i_k})\right\| + \sum_{j=i_1,\ldots,i_k} avg_j.$$

Thirdly, $\|\xi^{\rightarrow}(\mathcal{S}_{i_1,\ldots,i_k})\|$ should be incorporated in the branch-and-bound technique applied while searching any subspace $\mathcal{S}_{i_1,\ldots,i_k}$.

In short, all the changes concern solely the way the coalition values are incorporated in the algorithm. Operations on every subspace $\mathcal{S}_{i_1,\ldots,i_k}$ are based on $v_r(C)$ added to $\|\xi^{\rightarrow}(\mathcal{S}_{i_1,\ldots,i_k})\|$ and they do not affect any of the anytime properties of the algorithm. Very similar refinements can be made for any system of class $S_3^{\overleftarrow{E}}$ or S_3^T.

7.3 General Algorithm

Michalak *et al* showed that, for some particular settings with externalities, it is possible to bound the values of every coalition in all coalition structures in which this coalition is embedded. Consequently, they used a refined version of Rahwan *et al*'s algorithm. In many cases, this approach generated an optimal solution without having to search the entire space of coalition structures. We will now demonstrate how the algorithm of Michalak *et al* can be used to solve *any* multi-agent system in which bounds on externalities belong to category A or B (see Subsection 7.1). Let us assume that, in a given system represented by $PFG_{\overrightarrow{E}}$, a coalition C induces, in *every* coalition structure, a sum of outward operational externalities within the bounds $\overrightarrow{\mathcal{E}}_{LB}(C)$ and $\overrightarrow{\mathcal{E}}_{UB}(C)$ *i.e.*, $\overrightarrow{\mathcal{E}}_{LB}(C) \leq \left\|\overrightarrow{\mathcal{E}}(C,\Pi)\right\| \leq \overrightarrow{\mathcal{E}}_{UB}(C)$. Adding $v_r(C)$ to every element yields:

$$v_r(C) + \overrightarrow{\mathcal{E}}_{LB}(C) \leq v_r(C) + \left\|\overrightarrow{\mathcal{E}}(C,\Pi)\right\| \leq v_r(C) + \overrightarrow{\mathcal{E}}_{UB}(C)$$

By setting $LB(C) = v_r(C) + \overrightarrow{\mathcal{E}}_{LB}(C)$ and $UB(C) = v_r(C) + \overrightarrow{\mathcal{E}}_{UB}(C)$ and using formula (2) we get:

$$LB(C) \leq \mathcal{V}_{\overrightarrow{E}}(C,\Pi) \leq UB(C) \tag{20}$$

That is, the value of every embedded coalition has upper and lower bounds. Furthermore, these bounds are independent from Π, meaning that condition (20) can be interpreted as a partition function bounded by two (forms of) characteristic functions. Exactly two such functions are the input to Step 2 of Michalak *et al*'s algorithm where the pruning of the search space takes place. Of course, as demonstrated in Figure 10, bounds computed in the above way are not going to be, in general, as tight as those computed when there are no externalities. However, it is still possible, using these bounds, to prune several subspaces and save on computation time. Following similar reasoning, also $\mathcal{P}(C,\Pi)$ under $PFG_{\overleftarrow{E}}$ or PFG_T can be bounded provided sufficient information about externalities is available.

Finally, it should be stressed that the above algorithm is able to solve the coalition structure generation problem in both games with and without externalities. In the latter case, for a given characteristic function v, upper and lower bounds for any coalition C should be defined as $UB(C) = LB(C) = v(C)$.

8. CONCLUSIONS

This paper considered the issue of representing coalitional games with externalities. The conventional game-theoretic approach defines externalities as a result of mergers of two coalitions in the system. However, while such a definition is convenient in some applications, in others it is not. In particular, to solve the coalition structure generation problem in a cooperative environment, one does not have to consider mergers of coalitions that create coalition structures. To this end we referred to a notion of total externalities from coalition formation and, subsequently, proposed a related representation. In this representation, the total value of an externality that affects a coalition is independent of the way this coalition was formed. Furthermore, we propose a new notion of externalities that describes the effect that each coalition has on the entire system and *vice versa*. Building on this new notion, we propose another two representations and compare them to the game-theoretic approach. In particular, we show that the new representations are fully expressive and, for many classes of games, more concise than the conventional PFG. Building upon this insight, we propose a number of approaches to solve the coalition structure generation problem in systems with externalities. We show that, if externalities are characterised by various degrees of regularity, the new representations allow us to adapt some of the algorithms that were originally designed for domains with no externalities so that they can be used when externalities are present. Finally, building upon [16] and [9], we present a unified method to solve the coalition structure generation problem in any system, with or without externalities.

There are a number of directions for future research. An interesting one is to evaluate the performance of Michalak *et al.*'s algorithm for systems characterised by bounds of varying tightness. Furthermore, there are various computational issues related to other important game theoretical concepts. In particular, it would be very interesting to study how the new representations can be used to facilitate efficient calculation of the Shapley value and the core related questions in games with externalities.

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